

THE T^1 LIFTING CRITERION FOR UNOBSTRUCTED DEFORMATIONS (OF CALABI-YAUS)

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We present a simple cohomological criterion for a given compact complex manifold X to have unobstructed deformations. The criterion was introduced by Z. Ran [1] as a means to prove the Bogomolov-Tian-Todorov theorem ([3], [4])—originally proven by $\bar{\partial}$ -methods, this states that deformations of Calabi-Yau manifolds are unobstructed.

A simple deformation-theoretic proof of Ran’s lifting criterion is given, due to Kawamata [2], using the characterization of pro-representability of deformation functors ([7]).

1. DEFORMATIONS AND OBSTRUCTIONS

Let X be a nonsingular variety over $k = \bar{k}$. Let A be an Artin local algebra over k . A *deformation* Y of X over A is a scheme Y , flat over $\mathrm{Spec} A$, together with an identification

$$X = Y \times_{\mathrm{Spec} A} \mathrm{Spec} k$$

of X with the fiber over the closed point $\mathrm{Spec} k \hookrightarrow A$. Thus X is a closed subscheme of Y with the same underlying topological space. In the case $A = A_n = k[x]/x^{n+1}$, we say that Y is an n ’th-order deformation of X .

N. B. Throughout, it will be possible to consider an arbitrary compact complex manifold X without scheme structure, and deformations of its sheaf $\mathcal{O}_X^{an}$ of analytic functions.

The motivation for this is well-known to be the pullback of a flat family by a tangent vector at a point. Note that we can also make these definitions *relative* to a base S , so that a deformation comes from an S -scheme with an S -valued point and some S -scheme thickening it. We will just need to define a *first-order deformation of an S -scheme X* to be a scheme over $S \times_k k[\epsilon]$.

Example 1.1. (Potential confusion) Because $S_{n+1} = \mathrm{Spec} A_{n+1}$ is not an S_n -scheme, an $(n+1)$ -st order deformation *is not* an infinitesimal deformation relative to S_n . (It is, however, an infinitesimal extensions (thickenings) of it as deformations of X .) The distinction is addressed by the T^1 lifting criterion.

We will consider the *deformation functor* of the fixed scheme X , denoted $D_X : \mathcal{C} \rightarrow \text{Set}$.

Artin ring A	$\longrightarrow$	{Isomorphism classes of deformations over A of a fixed object X (e. g. subscheme, line bundle, scheme)} ¹
morphism	$\longrightarrow$	base change a deformation by that morphism.

1.1. (Infinitesimal) Lifting.

Definition 1.2. An *infinitesimal extension* is one of the form $0 \rightarrow I \rightarrow A' \rightarrow A \rightarrow 0$, with $I^2 = 0$. A small extension is one where $I = (t)$ is principal and $tm_{A'} = 0$. Note that any extension of Artin rings factors into small extensions (Nakayama).

Proposition 1.3. (*Infinitesimal Lifting Property*) *Exercise in Hartshorne, solution given in [5] 4.4.*

Proof. Note that this uses the vanishing ([5] 8.1) of H^1 of a quasi-coherent sheaf on an any affine scheme. For relative think of fiber product. Splitting is as A' -modules, so makes genuinely trivial as a ring. $\square$

1.1.1. Flatness.

Proposition 1.4. *Let $0 \rightarrow J \rightarrow A' \rightarrow A \rightarrow 0$ be an infinitesimal extension of noetherian rings. Then an A' -module M' is flat over A' if and only if*

- (1) $M = M' \otimes_{A'} A$ *is flat over A , and*
- (2) *the natural map $M \otimes_A J \rightarrow M'$ is injective.*

Proof. Note that we have the identity of A -modules $M' \otimes_{A'} J = M \otimes_A J$. This is [5] 2.2, second part uses Tor sequence / local criterion. $\square$

Lemma 1.5. ([7] Lemma 3.3) *A map of deformations over A that restricts to the identity on the closed fiber is an isomorphism.*

This could be phrased as “all morphisms in the category of deformations over A are invertible,” but we don’t refer to such a category here.

Corollary 1.6. *Any deformation of a nonsingular affine variety is trivial.*

Proof. Factor by infinitesimal extensions and trivialize step by step, to get map from $X \times_k A$ to X restricting to the identity on the closed fiber. By the flatness Lemma just proven, this is an isomorphism. $\square$

1.2. First-Order Deformations. We use the following here and again later:

Lemma 1.7. *Consider D -automorphisms of the trivial deformation that restrict to the identity on the closed fiber): $\text{Aut}_D^*(X \times_k D) = H^0(X, T_X)$. More generally, if $0 \rightarrow I \rightarrow A' \rightarrow A \rightarrow 0$ is an infinitesimal extension, Y a deformation over A' , then the automorphisms of Y restricting to the identity on $Y \times_{A'} A$ are given by $I \otimes_k H^0(X, T_X)$.*

Proof. (Note these are literally scheme automorphisms; of Y , a fixed representative of its isomorphism class of deformations.) $\square$

Theorem 1.8. $H^1(T_X)$ classifies first-order deformations of X . If $S = \text{Spec } A$ is an Artin ring and X' is a deformation of X over A , then first-order deformations of X' in the category of S -schemes are classified by $H^1(T_{X'/S})$.

Proof. We would think this is true, just from the Kodaira-Spencer map. To really prove it, one needs to trivialize over affines and make a cocycle argument based on the Lemma. $\square$

Remark 1.9. We can carry over all our arguments for first-order deformations to arbitrary infinitesimal deformations of X by considering e. g. $H^1(T_{X/S} \otimes \mathcal{F})$, where $\mathcal{F}$ is the coherent sheaf that we extend by. (This pulls out if it's a constant sheaf I and flat, as will always be our case—i. e. for deformations by a ring of infinitesimals $k[I] = k[I]/I^2$.)

Example 1.10. We want to give an example of a nontrivial infinitesimal deformation of smooth projective varieties. Consider the family of elliptic curves $E_\lambda = Y^2Z = (X + \lambda Z)(X + Z)(X - Z)$. (For big reasons, should know a lot about how this behaves). How to show that the infinitesimal deformation around $\lambda = 0$ is not trivial? (Question of whether j -map is ramified, or whether it's a moduli space at all.)

If we do this just for the affine family $f(x, y, \epsilon) = y^2 = (x + \epsilon)(x + 1)(x - 1)$, one can find a section

$$\begin{aligned} k[x, y] / f(x, y) &\longrightarrow k[x, y, \epsilon] / (f(x, y, \epsilon), \epsilon^2) \\ x &\mapsto x + (1 - \tfrac{3}{2}x^2)(x^2 - 1)\epsilon \\ y &\mapsto y + \tfrac{9}{4}xy(x^2 - 1)\epsilon. \end{aligned}$$

A better way to do this is to consider the co-normal sequence

$$0 \rightarrow \mathcal{O}_X \xrightarrow{d\epsilon} \Omega_{X_2} \otimes_{X_2} \mathcal{O}_X \rightarrow \Omega_X \rightarrow 0,$$

(the first map is injective by flatness). The element ϵ is in fact a global nilpotent element of the structure sheaf $\mathcal{O}_{X_2}$, and *on an affine* we just have $\Omega_{X_2} \stackrel{\text{loc}}{=} \Omega_X \otimes_k k[\epsilon]$, by Hartshorne 8.2(A) since the extension is trivial.

To see the extension is not trivial, one could literally compute the image of the boundary map ($\Omega_X \cong \mathcal{O}_X$). Or one can just examine the pullback of the form dx/y , of the form $\omega = dx/y + Ad\epsilon$. On the patch $g'(x) \neq 0$, ($y^2 = g(x)$), we get

$$\omega = 2\frac{dy}{g'} + (A + \frac{(x+1)(x-1)}{yg'})d\epsilon,$$

so we see that A has a single simple pole at $x = y = 0$ on the affine $x - y$ plane. But you actually get that A must be holomorphic at ∞ . So no such A is possible, i. e. the map is not surjective on global sections.

In fact, these extensions classify deformations, if X is smooth. (And thus one sees that $H^1(X, T_X) = \text{Ext}_X^1(\mathcal{O}_X, \Omega_X)$ classifies deformations.)

Remark 1.11. For non-smooth schemes, these extensions don't necessarily classify all deformations, and similar but more complicated functors are necessary (T^1 and T^2 functors / cotangent complex, see [8]).

1.3. Higher-Order Deformations and the Obstruction Group $H^2(T_X)$.

Higher-order deformations, i. e. elements (classes) in $D_X(A)$ for general $A \in \mathcal{C}$, are not in general classified by nice linear spaces. The remainder of the paper will therefore be occupied with the question of lifting deformations to higher order, i. e. determining the image $D(A') \rightarrow D(A)$ for a small extension of Artin rings $A' \twoheadrightarrow A$.

To start on this lifting problem, we need to make precise the following notion:

Definition 1.12. For a deformation functor D , we say that a k -vector-space T^2 is an *obstruction group* ([6]) if for any surjection $A' \rightarrow A$ we get a sequence

$$D(A') \xrightarrow{D(\phi)} D(A) \xrightarrow{\delta} T^2 \otimes I,$$

which is functorial in squares of rings with surjective rows. We'll actually always have a very elegant (but not at all transparent or geometrical) obstruction space.

Let $A' \in \mathcal{C}$ be an Artin local ring whose maximal ideal $\mathfrak{m}$ has square zero.

$$\begin{array}{ccccccc} 0 & \longrightarrow & I & \longrightarrow & \mathfrak{m} & \longrightarrow & \mathfrak{m}/I \longrightarrow 0 \\ 0 & \longrightarrow & I \otimes T_X & \longrightarrow & \mathfrak{m} \otimes T_X & \longrightarrow & \mathfrak{m}/I \otimes T_X \longrightarrow 0. \end{array}$$

1.3.1. Higher-Order Deformations. This very naive argument fails for deformations over arbitrary A because the automorphism group of even the trivial deformation over an affine one is not necessarily abelian—and so deformations over A may not be classified by some nice linear space.

Still, for a small extension by I and over affine X , we have a sort-of sequence of groups:

$$I \otimes_k T_X \rightarrow \text{Aut}_D^*(X \times_k A') \rightarrow \text{Aut}_D^*(X \times_k A) \rightarrow 0,"$$

meaning that we have an action of the left-hand group on each fiber of the map at right. Ideally, we'd want each fiber to be a torsor. The boundary map argument used to show the infinitesimal lifting obstruction actually works here as well, to show that 2-cocycles of derivations are the obstruction to lifting. (You have to literally do the boundary map and show it works, starting from the transition functions on trivializations—Johan.)

So one can actually conclude that $H^2(T_X)$ is the obstruction group to lifting of deformations, in our case.

Remark 1.13. The cocycle argument shows that small deformations of A' sitting above a fixed A have an action by $H^1(T_X) \otimes I = t_D \otimes I$, given by composing the “co-cycles” giving the data of a deformation with the fairly trivial ones where the derivations just take values in I . The question, of whether this action makes the set of extensions to A' of a fixed deformation over A into a torsor (or just pseudo-torsor) for $H^1(T_X) \otimes I$, is crucial for the study of higher-order deformations (see Proposition 2.6).

Remark 1.14. Note the importance of just checking small deformations.

Example 1.15. The product $A \times \mathbb{P}^n$, with A an Abelian variety of dimension greater than 1 and $n > 1$, has non-liftable first-order deformations. (Prof. Thaddeus).

2. SCHLESSINGER'S CRITERIA

To deal with higher-order deformations, we want to “take a limit” of these Artin rings over which we deform X , so we can classify all deformations of A by maps $\text{Spec } A \rightarrow \text{Spec } R$. (In fact there is a universal formal scheme containing X , from which all deformations are obtained by base change.)

Let k be a field,² and consider the category $\mathcal{C}$ of Artinian local k -algebras having residue field k , and local morphisms. We consider only covariant functors $D : \mathcal{C} \rightarrow \mathbf{Sets}$ such that $D(k)$ consists of one element.

Definition 2.1. Define the tangent space to a functor $t_D = D(k[\epsilon])$. This has a natural vector space structure and agrees for the functor D_X with the Kodaira-Spencer tangent space.

Definition 2.2. We say that a functor D is *pro-representable* if we have a natural isomorphism $h_R \rightarrow D$ for some $R \in \hat{\mathcal{C}}$, the set of complete Artinian local k -algebras all of whose R/m^n are in $\mathcal{C}$.

²or at least a Noetherian complete local algebra over a field

We say that D just has a *hull* if there is a smooth map of functors $h_R \rightarrow D$, i. e. for any surjection $B \rightarrow A$ in $\mathbb{C}$, the induced map

$$h_R(A) \rightarrow h_R(A) \times_{D(A)} D(B)$$

is surjective. (This definition seems actually to be motivated by the infinitesimal lifting property.)

Theorem 2.3. (Schlessinger's criteria) *Let D be a functor, let $A' \rightarrow A$ be morphisms in $\mathcal{C}$, and consider the map*

$$(1) \quad D(A' \times_A A'') \rightarrow D(A') \times_{D(A)} D(A'')$$

Assume the functor D satisfies the following:

(H₁) (1) is a surjection whenever $A'' \rightarrow A$ is a small (or for that matter, any) extension

(H₂) (1) is a bijection when $A = k$, $A'' = k[\epsilon]$ (or any ring of infinitesimals)

(H₃) $\dim_K(t_D) < \infty$

Then D has a hull, R . If, moreover,

(H₄) (1) is surjective whenever $A' = A''$ is a small extension, then D is pro-representable.

Proof. ([7]) The ring R is an inverse limit of successive quotients $k[[x_1, \dots, x_r]]/J$, where $r = \dim_K(t_D)$, and each successive quotient assures the required property of Artin rings of length one more. As the quotients get smaller w.r.t m_R , the inverse limit is again a quotient of R . $\square$

Definition 2.4. A functor satisfying (H_{1-3}) above is sometimes simply referred to as a *deformation functor*.

Corollary 2.5. *The deformation functor $D = D_X$ has a hull (in particular, a quotient of a power-series ring generated by a basis of t_D).*

Proposition 2.6. *Let R be the hull of D_X . Then (H_4) is satisfied if all automorphisms of a deformation $Y' \otimes_{A'} A$ lift to Y .*

Proof. ([7] 3.10) If you go through as in [7] and write down functorially the action of $H^1(X, T_X)$ on extensions of a fixed deformation, (H_4) is equivalent to the requirement that it's free, and so that the extensions form a torsor (possibly empty). One can then relate this to the automorphism question, a bit like in Galois theory. $\square$

This will be used later to show that the deformation functor of a Calabi-Yau manifold is pro-representable.

Proposition 2.7. *In general, one can define the obstruction group to be $(I/m_R I)^\sim$.*

Proof. Straightforward ([6]). $\square$

Example 2.8. ([5] ex 19.2) The deformation functor of the Hirzebruch surface $\mathbb{F}_2$ is not pro-representable.

3. COMPARISON TO NOT-FORMAL MODULI QUESTIONS?

It looked to me like there might be an analogy (a forgetful functor, even?) of the form “hull $\leftrightarrow$ coarse moduli space, pro-representable $\leftrightarrow$ fine moduli space,” but in fact pro-representable just corresponds to the completion of the local ring of a coarse moduli stack under certain conditions.

4. THE T^1 LIFTING CRITERION FOR UNOBSTRUCTEDNESS

Definition 4.1. Let X be a compact complex manifold. We say that X (or rather its deformation functor D_X) satisfies the T^1 *lifting property* if for each k and for any commutative diagram

$$\begin{array}{ccc} X_k & \subset & X_{k+1} \\ \downarrow & & \downarrow \\ S_k & \subset & S_{k+1} \end{array}$$

of deformations of X , the map $H^1(T_{X_{k+1}/S_{k+1}}) \rightarrow H^1(T_{X_k/S_k})$ is surjective.

Remark 4.2. “The map” $T_{X_{k+1}/S_{k+1}} \rightarrow T_{X_k/S_k}$ is given just by post-composing a derivation with the base change $\mathcal{O}_{X_{k+1}} \rightarrow \mathcal{O}_{X_k}$, which descends to a derivation on X_{k+1} . (The ordinary inclusion of tangent spaces goes the wrong way.)

Define rings $B_n = A_n \otimes A_1$, which has a natural inclusion of and projection to A_n .

Definition 4.3. For any deformation functor D and $X_k \in D(A_k)$, we can define

$$T^1(X_k/A_k) = \{Y_k \in D(B_k) \mid D(\beta_k)(Y_k) = X_k\}.$$

We say that D satisfies the T^1 *lifting property* if for each k the natural homomorphism

$$T^1(\alpha_k) : T^1(X_{k+1}/A_{k+1}) \xrightarrow{D(\alpha_n)^\circ} T^1(X_k/A_k)$$

is surjective for $X_k = D(\alpha)(X_{k+1})$.

Remark 4.4. We saw above that for $D = D_X$, X smooth, this definition of $T^1(X_k/A_k)$ is just the functor of “first-order deformations of X_k relative to S_k ,” as discussed above: as we say, this is given by $H^1(T_{X_k/S_k})$. So the two definitions of T^1 lifting correspond. This is equivalent to the T^1 lifting property for D_X , as follows: (*)

Remark 4.5. The functor T^1 and the functor T^2 defined below can be constructed more generally for singular varieties using the cotangent complex [8], and the next criterion holds as long as T^1 coincides with definition 4.3.

Theorem 4.6. (T^1 Lifting Criterion) *Let D be a deformation functor pro-represented by a ring R . Assume the base field k has characteristic zero, and D has an obstruction space T^2 . Then R is smooth over k (in fact a power-series ring), and D is unobstructed.*

Remark 4.7. The hypothesis that R in fact pro-represents D , rather than just being a hull, is necessary for the following argument of Kawamata [2]. However, it certainly can be loosened [6], and can be dispensed with entirely in the case $D = D_X$ for X a smooth complex manifold [1].

Proof. From the proof of Schlessinger’s criterion for pro-representability, we may write $R = k[[x_1, \dots, x_r]]/I$, with $I \subset m_R^2$ and $r = \dim(t_D)$. We will prove that $I = 0$. In fact, for this we do not need surjectivity of D in all cases, but only

$$D(\alpha_n) : D(A_{n+1}) \twoheadrightarrow D(A_n)$$

for each n . This is seen as follows: given any $a_1, \dots, a_r \in k$, we can build a map

$$\begin{array}{ccc} R & \xrightarrow{x_i \mapsto a_i t} & A_2 = k[t]/t^2 \\ & \searrow & \uparrow \\ & & k[t]/t^3 \\ & \searrow & \uparrow \\ & & \vdots \\ & \searrow & \uparrow \\ & & k[[t]] \end{array}$$

where initially we use $I \subset m_R^2$, and then lift to the inverse limit. (In other words, there is a formal curve through any given tangent vector at the origin of the formal spectrum of R .) Now assume I contains a nonzero power series $f = f_d + f_{d+1} + \dots$, $d > 0$, so we could choose a_i such that $f_d(a_1, \dots, a_r) \neq 0$. But then the above map sends f to $t^d f_d(a_1, \dots, a_r) + t^{d+1} f_{d+1} + \dots \neq 0 \in k[[t]]$, a contradiction. Thus for R to be a power series ring, it sufficed to verify lifting of $D(\alpha_n)$.

Here’s a rough explanation of the next argument. We have the idea to consider lifting these first-order deformations between a *given* pair of $n - 1$ ’st and n ’th order deformations—we would already be considering this

just because they are classified by the very standard functor T^1 (in our case a nice H^1 group). We want to know if the obstruction vanishing for these infinitesimals will imply that the obstruction to lifting will actually vanish overall. So we could prove that the T^1 lifting question actually “sees” the obstruction to lifting an n ’th to an $n + 1$ ’st order deformation.

The naive idea would be to extend the deformation in the infinitesimal direction of B_n , before extending to A_{n+1} . This doesn’t make sense. One actually has to introduce a new ring C_n , slightly smaller than B_n , for which one has a map $A_n \rightarrow C_n$ which gives a non-trivial base change. This could be described as “deforming in a diagonal direction,” that of the long arrow in the following diagram. This is not really “intermediate” to extending to A_n ; however, this base change actually induces an isomorphism (φ in diagram 2) between the obstructions to extending deformations over A_n to A_{n+1} and the obstruction to extending from base C_n to B_n . And this latter question is controlled by T^1 lifting. A schematic:

$$\begin{array}{ccccccc}
 X & \rightrightarrows & X_1 & \longrightarrow & \cdots & \longrightarrow & X_{n-1} \\
 & & & & & & \downarrow \\
 & & & & & & Z \dashrightarrow Y
 \end{array}
 \quad
 \begin{array}{ccc}
 & \xleftarrow{A_n} & X_n \\
 & \downarrow & \downarrow \\
 & C_n & B_n
 \end{array}
 \quad
 \begin{array}{ccc}
 & \xleftarrow{A_{n+1}} & ? \\
 & \downarrow & \downarrow \\
 & B_n & ?
 \end{array}$$

So we’re actually able to “peek around” the obstruction by sidestepping infinitesimally, while taking the deformation X_n with us, provided we can still peek out in the same range of directions after extending from X_{n-1} to X_n (which is T^1 lifting).

Define the ring $C_n = B_n/(x^n y) = k[x, y]/(x^{n+1}, x^n y, y^2)$, and note that $C_n = B_{n-1} \times_{A_{n-1}} A_n$. Define maps

$$\epsilon_n : A_{n+1} \rightarrow B_n$$

$$\epsilon'_n : A_n \rightarrow C_n$$

$$t \mapsto x + y.$$

Note that $(x + y)^{n+1} = x^{n+1} + (n + 1)x^n y + y^2(\cdots)$, so these are defined and the first has a kernel (t^{n+1}) .

The relevant rings form a commutative diagram, with all maps the natural ones except ϵ_n, ϵ'_n ,

$$\begin{array}{ccccc}
 A_{n+1} & \longrightarrow & A_n & \xrightarrow{\quad} & A_{n-1} \\
 & \searrow \epsilon_n & \uparrow & \swarrow \epsilon'_n & \uparrow \\
 & & B_n & \xrightarrow{\quad} & B_{n-1}
 \end{array}
 \quad
 \begin{array}{ccc}
 & & C_n \\
 & \swarrow & \searrow \\
 & B_n & B_{n-1}
 \end{array}$$

where the left square is a fiber product of rings (sets), i. e. $C_n = B_{n-1} \times_{A_{n-1}} A_n$.

Now, if we apply D to the diagram, we get a fiber square on the right because of Schlessinger's condition (H_4) . The key diagram comes from the definition and functoriality of the obstruction group T^2 :

$$(2) \quad \begin{array}{ccccc} D(A_{n+1}) & \xrightarrow{D(\alpha_n)} & D(A_n) & \xrightarrow{\delta_1} & T^2 \otimes (t^{n+1}) \\ \downarrow D(\epsilon_n) & & \downarrow D(\epsilon'_n) & & \downarrow \varphi \\ D(B_n) & \xrightarrow{D(\gamma_n)} & D(C_n) & \xrightarrow{\delta_2} & T^2 \otimes (x^n y), \end{array}$$

where the rows are "exact."

Since φ is an isomorphism, it suffices to show that δ_2 is zero, for then δ_1 is zero and the obstruction vanishes as claimed. In fact the T^1 lifting hypothesis implies that $D(\gamma_n)$ is surjective. Let $Z_n \in D(C_n)$. On the right edge of the fiber square, the image of Y_n gives an element of $T^1(X_{n-1}/A_{n-1})$ exactly by definition 4.3. Applying the T^1 lifting hypothesis to the image of Z_n in $D(A_n)$, we get an element $Y_n \in T^1(X_n/A_n) \subset D(B_n)$ that lifts this element of $T^1(X_{n-1}/A_{n-1})$. Since C_n is in fact the fiber product, we get that the image of Y_n must be Z_n . Therefore the bottom left arrow of 2 is surjective, and we are done.

Finally, since R is a power-series ring over k , $D = h_R$ is clearly unobstructed for lifting of *any* maps to Artin rings in $\mathcal{C}$ (Artin rings are complete). $\square$

Note. There was no induction.

This criterion is used for various kinds of deformation functors.

5. CONSEQUENCES OF UNOBSTRUCTEDNESS

The Kuranishi space of X is smooth.

6. APPLICATION TO CALABI-YAU'S

Theorem 6.1. *Let X be a complex manifold with $K_X = \mathcal{O}_X$ and $H^0(X, T_X) = 0$ (i. e. X has no infinitesimal automorphisms). Then X has unobstructed deformations.*

Proof. By Proposition 1.7 and induction, in fact no deformation has automorphisms preserving X . So by Proposition 2.6 the hull represents D_X . Hence we may use (our proof of) the T^1 lifting criterion.

To show the T^1 lifting, return to the definition. By ([9], Prop 5.5), the Hodge-to-de-Rham spectral sequence relative to the base S_k degenerates,

and each of the groups $H^i(\Omega_{X_k/S_k}^j)$ is a free $\mathcal{O}_{S_k}$ -module of rank $h^{i,j}$. Therefore K_{X_k/S_k} is free over a constant sheaf, hence trivial for all $k > 0$. So we may write

$$T_{X_k/S_k} = \Omega_{X_k/S_k}^{n-1},$$

and the relevant T^1 groups are free so we have the required surjections. $\square$

Remark 6.2. The theorem holds without the restriction about automorphisms, because of the decomposition theorem. (Or because the original proofs didn't use T^1 .)

Proposition 6.3. *Deformations of Calabi-Yau's stay Calabi-Yau.*

Corollary 6.4. *A versal deformation space in a category (e. g. the Kuranishi space, in the analytic category) is smooth in some maximal neighborhood.*

Proof. The completions of the local rings are the same. $\square$

Example 6.5. For quintic threefolds the obstruction group doesn't vanish (it has dimension 101), but we now know that the obstructions themselves vanish.

Remark 6.6. Infinitesimal Torelli holds, but global structure is open (known to be quasi-projective, however).

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